$$\text{Area} \approx \frac{\pi}{4}\left[\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi\right]$$
$$= \frac{\pi}{4}\left[\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right]$$
$$= \frac{\sqrt{2}+1}{4}\pi \approx 1.8961$$

For the figure on the right, each rectangle has width $\frac{\pi}{6}$.

$$\text{Area} \approx \frac{\pi}{6}\left[\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \frac{5\pi}{6} + \sin\pi\right]$$
$$= \frac{\pi}{6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right]$$
$$= \frac{\sqrt{3}+2}{6}\pi \approx 1.9541$$

(b) You could obtain a more accurate approximation by using more rectangles. You will learn later that the exact area is 2.

9. (a) $\text{Area} \approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

$\text{Area} \approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. Answers will vary. Sample *answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.2000	0.2000	0.1996	0.1961

$\lim_{x\to4}\frac{x-4}{x^2-3x-4} \approx 0.2000 \qquad \left(\text{Actual limit is } \frac{1}{5}.\right)$

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to2}\frac{x-2}{x^2-4} \approx 0.25 \qquad \left(\text{Actual limit is } \frac{1}{4}.\right)$

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3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2050	0.2042	0.2041	0.2041	0.2040	0.2033

$\lim_{x\to 0} \frac{\sqrt{x+6}-\sqrt{6}}{x} \approx 0.2041$ $\left(\text{Actual limit is } \frac{1}{2\sqrt{6}}.\right)$

4.

x	-5.1	-5.01	-5.001	-4.999	-4.99	-4.9
$f(x)$	-0.1662	-0.1666	-0.1667	-0.1667	-0.1667	-0.1671

$\lim_{x\to -5} \frac{\sqrt{4-x}-3}{x+5} \approx -0.1667$ $\left(\text{Actual limit is } -\frac{1}{6}.\right)$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625$ $\left(\text{Actual limit is } -\frac{1}{16}.\right)$

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$\lim_{x\to 4} \frac{[x/(x+1)]-(4/5)}{x-4} \approx 0.04$ $\left(\text{Actual limit is } \frac{1}{25}.\right)$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

8.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

9.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

10.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	1.1111	1.0101	1.0010	0.9990	0.9901	0.9091

$\lim_{x\to -3} \frac{x+3}{x^2+7x+12} \approx 1.0000$ (Actual limit is 1.)

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11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4 - 1}{x^6 - 1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

12.

x	−2.1	−2.01	−2.001	−1.999	−1.99	−1.9
$f(x)$	12.6100	12.0601	12.0060	11.9940	11.9401	11.4100

$\lim_{x\to -2} \frac{x^3 + 8}{x + 2} \approx 12.0000$ (Actual limit is 12.)

13.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000$ (Actual limit is 2.) (Make sure you use radian mode.)

14.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

15. $\lim_{x\to 3} (4 - x) = 1$

16. $\lim_{x\to 1} (x^2 + 3) = 4$

17. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (4 - x) = 2$

18. $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2 + 3) = 4$

19. $\lim_{x\to 2} \frac{|x - 2|}{x - 2}$ does not exist.

For values of x to the left of 2, $\frac{|x - 2|}{(x - 2)} = -1$, whereas

for values of x to the right of 2, $\frac{|x - 2|}{(x - 2)} = 1$.

20. $\lim_{x\to 5} \frac{2}{x - 5}$ does not exist because the function increases and decreases without bound as x approaches 5.

21. $\lim_{x\to 1} \sin \pi x = 0$

22. $\lim_{x\to 0} \sec x = 1$

23. $\lim_{x\to 0} \cos(1/x)$ does not exist because the function oscillates between −1 and 1 as x approaches 0.

24. $\lim_{x\to \pi/2} \tan x$ does not exist because the function increases without bound as x approaches $\frac{\pi}{2}$ from the left and decreases without bound as x approaches $\frac{\pi}{2}$ from the right.

25. (a) $f(1)$ exists. The black dot at $(1, 2)$ indicates that $f(1) = 2$.

(b) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x\to 4} f(x) = 2$.

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26. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at -2.

(b) $\lim_{x\to -2} f(x)$ does not exist. As x approaches -2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x\to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x\to 2} f(x) = \frac{1}{2}$.

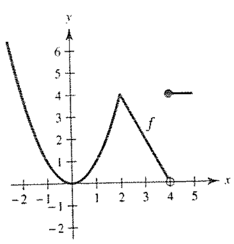
(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

(h) $\lim_{x\to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

27. $\lim_{x\to c} f(x)$ exists for all $c \neq -3$.

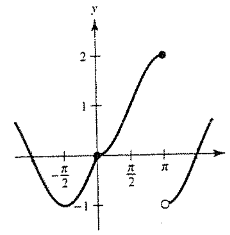
28. $\lim_{x\to c} f(x)$ exists for all $c \neq -2, 0$.

29.



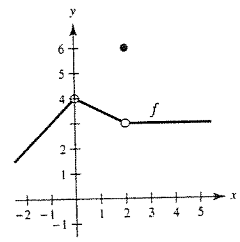
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

30.

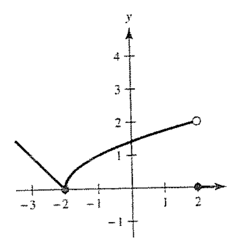


$\lim_{x\to c} f(x)$ exists for all values of $c \neq \pi$.

31. One possible answer is

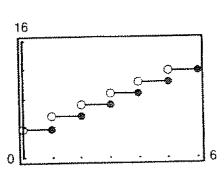


32. One possible answer is



33. $C(t) = 9.99 - 0.79[[-(t - 1)]]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	11.57	12.36	12.36	12.36	12.36	12.36	12.36

$\lim_{t\to 3.5} C(t) = 12.36$

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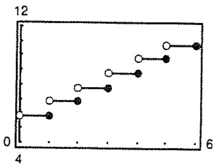
(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	10.78	11.57	11.57	11.57	12.36	12.36	12.36

The $\lim_{t\to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

34. $C(t) = 5.79 - 0.99[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	7.77	8.76	8.76	8.76	8.76	8.76	8.76

$\lim_{t\to 3.5} C(t) = 8.76$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	6.78	7.77	7.77	7.77	8.76	8.76	8.76

The limit $\lim_{t\to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

35. You need $|f(x) - 3| = |(x+1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x+1) - 3| = |f(x) - 3| < 0.4$, as desired.

36. You need $|f(x) - 1| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < |x - 2| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$

$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$

$$\Rightarrow |x - 1| > \frac{100}{101}$$

and you have

$$|f(x) - 1| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

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37. You need to find δ such that $0 < |x - 1| < \delta$ implies

$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$\begin{aligned} -0.1 &< \frac{1}{x} - 1 < 0.1 \\ 1 - 0.1 &< \frac{1}{x} < 1 + 0.1 \\ \frac{9}{10} &< \frac{1}{x} < \frac{11}{10} \\ \frac{10}{9} &> x > \frac{10}{11} \\ \frac{10}{9} - 1 &> x - 1 > \frac{10}{11} - 1 \\ \frac{1}{9} &> x - 1 > -\frac{1}{11}. \end{aligned}$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$\begin{aligned} -\frac{1}{11} &< x - 1 < \frac{1}{11} \\ -\frac{1}{11} &< x - 1 < \frac{1}{9}. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

38. You need to find δ such that $0 < |x - 2| < \delta$ implies

$|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 &< x^2 - 4 < 0.2 \\ 4 - 0.2 &< x^2 < 4 + 0.2 \\ 3.8 &< x^2 < 4.2 \\ \sqrt{3.8} &< x < \sqrt{4.2} \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$\begin{aligned} -\left(\sqrt{4.2} - 2\right) &< x - 2 < \sqrt{4.2} - 2 \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

39. $\lim_{x \to 2}(3x + 2) = 8 = L$

$$\begin{aligned} |(3x + 2) - 8| &< 0.01 \\ |3x - 6| &< 0.01 \\ 3|x - 2| &< 0.01 \\ 0 < |x - 2| &< \tfrac{0.01}{3} \approx 0.0033 = \delta \end{aligned}$$

So, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$\begin{aligned} 3|x - 2| &< 0.01 \\ |3x - 6| &< 0.01 \\ |(3x + 2) - 8| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

40. $\lim_{x \to 4}\left(4 - \frac{x}{2}\right) = 2 = L$

$$\begin{aligned} \left|\left(4 - \frac{x}{2}\right) - 2\right| &< 0.01 \\ \left|2 - \frac{x}{2}\right| &< 0.01 \\ \left|-\frac{1}{2}(x - 4)\right| &< 0.01 \\ 0 < |x - 4| &< 0.02 = \delta \end{aligned}$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\begin{aligned} \left|-\frac{1}{2}(x - 4)\right| &< 0.01 \\ \left|2 - \frac{x}{2}\right| &< 0.01 \\ \left|\left(4 - \frac{x}{2}\right) - 2\right| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

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41. $\lim_{x\to 2}(x^2 - 3) = 1 = L$

$$\left|(x^2 - 3) - 1\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|(x + 2)(x - 2)\right| < 0.01$$
$$\left|x + 2\right|\left|x - 2\right| < 0.01$$
$$\left|x - 2\right| < \frac{0.01}{\left|x + 2\right|}$$

If you assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

So, if $0 < \left|x - 2\right| < \delta = 0.002$, you have

$$\left|x - 2\right| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{\left|x + 2\right|}(0.01)$$
$$\left|x + 2\right|\left|x - 2\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|(x^2 - 3) - 1\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

42. $\lim_{x\to 5}(x^2 + 4) = 29 = L$

$$\left|(x^2 + 4) - 29\right| < 0.01$$
$$\left|x^2 - 25\right| < 0.01$$
$$\left|(x + 5)(x - 5)\right| < 0.01$$
$$\left|x - 5\right| < \frac{0.01}{\left|x + 5\right|}$$

If you assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

So, if $0 < \left|x - 5\right| < \delta = \frac{0.01}{11}$, you have

$$\left|x - 5\right| < \frac{0.01}{11} < \frac{1}{\left|x + 5\right|}(0.01)$$
$$\left|x - 5\right|\left|x + 5\right| < 0.01$$
$$\left|x^2 - 25\right| < 0.01$$
$$\left|(x^2 + 4) - 29\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

43. $\lim_{x\to 4}(x + 2) = 6$

Given $\varepsilon > 0$:

$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|x - 4\right| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < \left|x - 4\right| < \delta = \varepsilon$, you have

$$\left|x - 4\right| < \varepsilon$$
$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

44. $\lim_{x\to -3}(2x + 5) = -1$

Given $\varepsilon > 0$:

$$\left|(2x + 5) - (-1)\right| < \varepsilon$$
$$\left|2x + 6\right| < \varepsilon$$
$$2\left|x + 3\right| < \varepsilon$$
$$\left|x + 3\right| < \frac{\varepsilon}{2} = \delta$$

So, let $\delta = \varepsilon/2$.

So, if $0 < \left|x + 3\right| < \delta = \frac{\varepsilon}{2}$, you have

$$\left|x + 3\right| < \frac{\varepsilon}{2}$$
$$\left|2x + 6\right| < \varepsilon$$
$$\left|(2x + 5) - (-1)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

45. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}\left|x - (-4)\right| < \varepsilon$$
$$\left|x - (-4)\right| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < \left|x - (-4)\right| < \delta = 2\varepsilon$, you have

$$\left|x - (-4)\right| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

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46. $\lim_{x\to 1}\left(\frac{2}{5}x + 7\right) = \frac{2}{5}(1) + 7 = \frac{37}{5}$

Given $\varepsilon > 0$:

$$\left|\left(\frac{2}{5}x + 7\right) - \frac{37}{5}\right| = \left|\frac{2}{5}x - \frac{2}{5}\right| < \varepsilon$$
$$\frac{2}{5}|x - 1| < \varepsilon$$
$$|x - 1| < \frac{5}{2}\varepsilon$$

So, let $\delta = \frac{5}{2}\varepsilon$.

So, if $0 < |x - 1| < \delta = \frac{5}{2}\varepsilon$, you have

$$|x - 1| < \frac{5}{2}\varepsilon$$
$$\left|\frac{2}{5}x - \frac{2}{5}\right| < \varepsilon$$
$$\left|\left(\frac{2}{5}x + 7\right) - \frac{37}{5}\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

47. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3 - 3| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|3 - 3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

48. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $|-1 - (-1)| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

49. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x - 0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

50. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

51. $\lim_{x\to -5}|x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\left||x - 5| - 10\right| < \varepsilon$

$$|-(x - 5) - 10| < \varepsilon \quad (x - 5 < 0)$$
$$|-x - 5| < \varepsilon$$
$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$
$$|-(x - 5) - 10| < \varepsilon$$
$$\left||x - 5| - 10\right| < \varepsilon \quad \text{(because } x - 5 < 0\text{)}$$
$$|f(x) - L| < \varepsilon.$$

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52. $\lim_{x \to 6} |x - 6| = |6 - 6| = 0$

Given $\varepsilon > 0$: $||x - 6| - 0| < \varepsilon$

$$|x - 6| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - 6| < \delta = \varepsilon$, you have

$$|x - 6| < \varepsilon$$
$$||x - 6| - 0| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

53. $\lim_{x \to 1} (x^2 + 1) = 2$

Given $\varepsilon > 0$:

$$|(x^2 + 1) - 2| < \varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$|(x + 1)(x - 1)| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \frac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$|(x^2 + 1) - 2| < \varepsilon$$
$$|f(x) - 2| < \varepsilon.$$

54. $\lim_{x \to -3} (x^2 + 3x) = 0$

Given $\varepsilon > 0$:

$$|(x^2 + 3x) - 0| < \varepsilon$$
$$|x(x + 3)| < \varepsilon$$
$$|x + 3| < \frac{\varepsilon}{|x|}$$

If you assume $-4 < x < -2$, then $\delta = \varepsilon/4$.

So for $0 < |x - (-3)| < \delta = \frac{\varepsilon}{4}$, you have

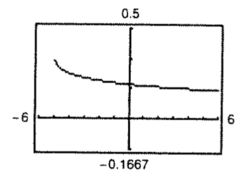
$$|x + 3| < \frac{1}{4}\varepsilon < \frac{1}{|x|}\varepsilon$$
$$|x(x + 3)| < \varepsilon$$
$$|x^2 + 3x - 0| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

55. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} 4 = 4$

56. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} x = \pi$

57. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

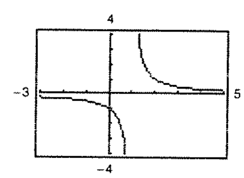
$\lim_{x \to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

58. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

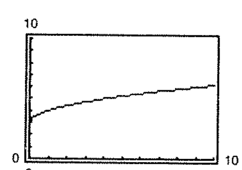
$\lim_{x \to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

59. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x \to 9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

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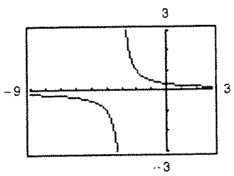
60. $f(x) = \dfrac{x-3}{x^2-9}$

$\lim_{x\to 3} f(x) = \dfrac{1}{6}$

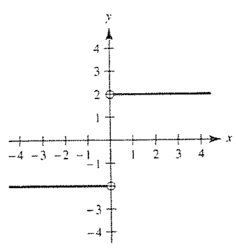


The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{6}\right)$.

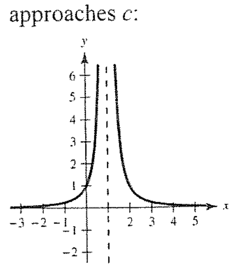
61. $\lim_{x\to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

62. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

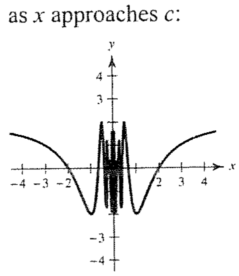
63. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



64. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

65. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x\to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

66. $V = \dfrac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

$r \approx 0.8397$ in.

(b) $2.45 \le V \le 2.51$

$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$

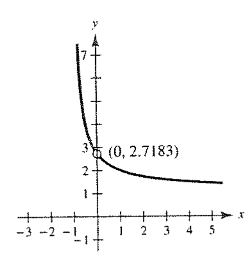
$$0.5849 \le r^3 \le 0.5992$$

$$0.8363 \le r \le 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

67. $f(x) = (1 + x)^{1/x}$

$\lim_{x\to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
−0.1	2.867972
−0.01	2.731999
−0.001	2.719642
−0.0001	2.718418
−0.00001	2.718295
−0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

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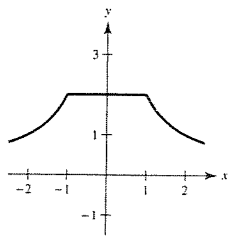
68. $f(x) = \dfrac{|x+1| - |x-1|}{x}$

x	-1	-0.5	-0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

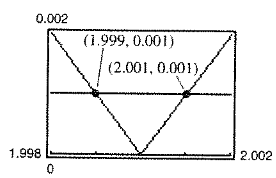
$\lim_{x \to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \neq 0, f(x) = \dfrac{(x+1) + (x-1)}{x} = 2.$



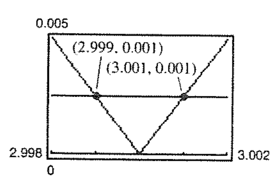
69.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

70.



From the graph, $\delta = 0.001$. So $(3 - \delta, 3 + \delta) = (2.999, 3.001)$.

Note: $\dfrac{x^2 - 3x}{x - 3} = x$ for $x \neq 3$.

71. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

72. True

73. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim_{x \to 2} f(x) = \lim_{x \to 2}(x - 4) = 2 \neq 0$

74. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x \to 2} f(x) = \lim_{x \to 2}(x - 4) = 2$ and $f(2) = 0 \neq 2$

75. $f(x) = \sqrt{x}$

$\lim_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

76. $f(x) = \sqrt{x}$

$\lim_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

77. Using a graphing utility, you see that

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim_{x \to 0} \dfrac{\sin 2x}{x} = 2$, etc.

So, $\lim_{x \to 0} \dfrac{\sin nx}{x} = n$.

78. Using a graphing utility, you see that

$\lim_{x \to 0} \dfrac{\tan x}{x} = 1$

$\lim_{x \to 0} \dfrac{\tan 2x}{x} = 2$, etc.

So, $\lim_{x \to 0} \dfrac{\tan(nx)}{x} = n$.

79. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

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80. $f(x) = mx + b, m \neq 0$. Let $\varepsilon > 0$ be given. Take

$\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

81. $\lim_{x \to c}[f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$,

then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when

$0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

82. (a) $$(3x + 1)(3x - 1)x^2 + 0.01 = (9x^2 - 1)x^2 + \frac{1}{100}$$
$$= 9x^4 - x^2 + \frac{1}{100}$$
$$= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)$$

So, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive. You can verify this with a graphing utility.p

(b) You are given $\lim_{x \to c} g(x) = L > 0$. Let

$\varepsilon = \dfrac{1}{2}L$. There exists $\delta > 0$ such that

$0 < |x - c| < \delta$ implies that

$|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \neq c$, you have $g(x) > \dfrac{L}{2} > 0$, as desired.

83. Answers will vary.

84. $\lim_{x \to 4} \dfrac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

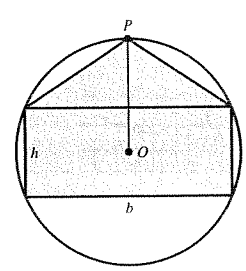
85. The radius OP has a length equal to the altitude z of the triangle plus $\dfrac{h}{2}$. So, $z = 1 - \dfrac{h}{2}$.

Area triangle $= \dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right)$

Area rectangle $= bh$

Because these are equal, $\dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$
$$\frac{5}{2}h = 1$$
$$h = \frac{2}{5}.$$



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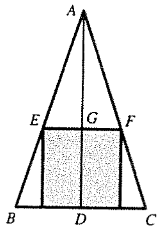
86. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$. Let x be the length of a side of the cube. Then $EF = x\sqrt{2}$.

By similar triangles,



$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3-x}{3}$$

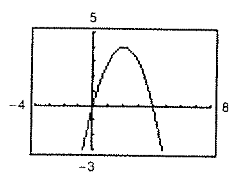
Solving for x, $3\sqrt{2}x = 6 - 2x$

$$\left(3\sqrt{2} + 2\right)x = 6$$

$$x = \frac{6}{3\sqrt{2}+2} = \frac{9\sqrt{2}-6}{7} \approx 0.96.$$

Section 1.3 Evaluating Limits Analytically

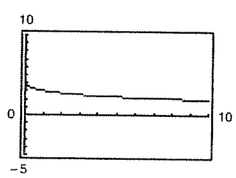
1.



(a) $\lim_{x\to 4} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = -5$

2.

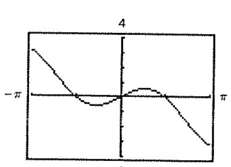


$$g(x) = \frac{12\left(\sqrt{x} - 3\right)}{x - 9}$$

(a) $\lim_{x\to 4} g(x) = 2.4$

(b) $\lim_{x\to 0} g(x) = 4$

3.



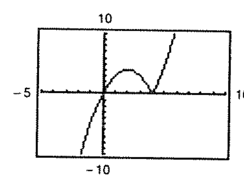
$f(x) = x\cos x$

(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524$

$\left(= \frac{\pi}{6}\right)$

4.



$f(t) = t\left|t - 4\right|$

(a) $\lim_{t\to 4} f(t) = 0$

(b) $\lim_{t\to -1} f(t) = -5$

5. $\lim_{x\to 2} x^3 = 2^3 = 8$

6. $\lim_{x\to -2} x^4 = (-2)^4 = 16$

7. $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

8. $\lim_{x\to -3} (3x + 2) = 3(-3) + 2 = -7$

9. $\lim_{x\to -3} \left(x^2 + 3x\right) = (-3)^2 + 3(-3) = 9 - 9 = 0$

10. $\lim_{x\to 1} \left(-x^2 + 1\right) = -(1)^2 + 1 = 0$

11. $\lim_{x\to -3} \left(2x^2 + 4x + 1\right) = 2(-3)^2 + 4(-3) + 1$
$= 18 - 12 + 1 = 7$

12. $\lim_{x\to 1} \left(3x^3 - 2x^2 + 4\right) = 3(1)^3 - 2(1)^2 + 4 = 5$

13. $\lim_{x\to 3} \sqrt{x+1} = \sqrt{3+1} = 2$

14. $\lim_{x\to 4} \sqrt[3]{x+4} = \sqrt[3]{4+4} = 2$

15. $\lim_{x\to -4} (x + 3)^2 = (-4 + 3)^2 = 1$

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