Practice Problems Week of April 6-10

#1

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

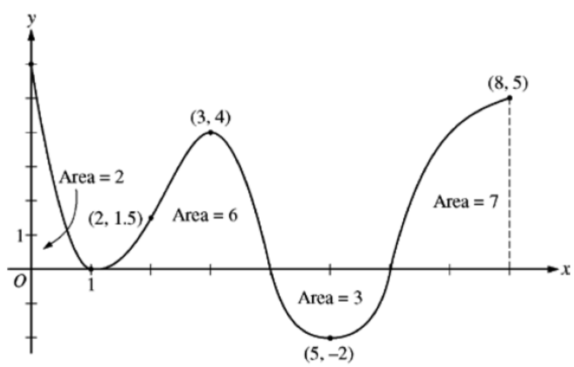
(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

#2

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

#3

Consider the differential equation $\frac{dy}{dx} = e^y\left(3x^2 - 6x\right)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

Answers

#1

(a) $C'(3.5) \approx \dfrac{C(4)-C(3)}{4-3} = \dfrac{12.8-11.2}{1} = 1.6$ ounces/min

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval)

$\dfrac{C(4)-C(2)}{4-2} = \dfrac{12.8-8.8}{2} = 2$

Therefore, by the Mean Value Theorem, there is at least one time t, $2 < t < 4$, for which $C'(t) = 2$.

(c) $$\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6}[2\cdot C(1) + 2\cdot C(3) + 2\cdot C(5)]$$
$$= \frac{1}{6}(2\cdot 5.3 + 2\cdot 11.2 + 2\cdot 13.8)$$
$$= \frac{1}{6}(60.6) = 10.1 \text{ ounces}$$

$\frac{1}{6}\int_0^6 C(t)\,dt$ is the average amount of coffee in the cup, in ounces, over the time interval $0 \le t \le 6$ minutes.

#2

(a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$f(0) = f(8) + \int_8^0 f'(x)\,dx$$
$$= f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$$
$$f(6) = f(8) + \int_8^6 f'(x)\,dx$$
$$= f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$
$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$$

#3

(a) $$\left.\frac{dy}{dx}\right|_{(x,\,y)=(1,\,0)} = e^0\left(3\cdot 1^2 - 6\cdot 1\right) = -3$$

An equation for the tangent line is $y = -3(x-1)$.

$f(1.2) \approx -3(1.2-1) = -0.6$

(b) $$\frac{dy}{e^y} = \left(3x^2 - 6x\right)dx$$
$$\int \frac{dy}{e^y} = \int \left(3x^2 - 6x\right)dx$$
$$-e^{-y} = x^3 - 3x^2 + C$$
$$-e^{-0} = 1^3 - 3\cdot 1^2 + C \Rightarrow C = 1$$
$$-e^{-y} = x^3 - 3x^2 + 1$$
$$e^{-y} = -x^3 + 3x^2 - 1$$
$$-y = \ln\left(-x^3 + 3x^2 - 1\right)$$
$$y = -\ln\left(-x^3 + 3x^2 - 1\right)$$

Note: This solution is valid on an interval containing $x = 1$ for which $-x^3 + 3x^2 - 1 > 0$.